

ON THE DIFFERENTIABLE PINCHING PROBLEM

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(Received August 31, 1967)

Introduction

Let M be a compact, connected, simply connected Riemannian manifold with a metric d and denote by K the sectional curvature of M . Then it is known that if K satisfies the following inequality:

$$1/4 < K \leq 1,$$

there exists a homeomorphism h of M onto S^n , the standard unit n -sphere ([1, 4, 6, 8]).

On the other hand, we also know that there is defined a positive $I(h)$ (≥ 1) for a homeomorphism h between two compact Riemannian manifolds, such that if $I(h)$ is sufficiently near to unity, that is, $(1 \leq) I(h) < 1 + \varepsilon(n)$ ($\varepsilon(n)$ is a positive depending on n), then h is approximated arbitrarily by diffeomorphisms ([5]).

Our main aim in the note is to investigate a relation between $I(h)$ and the sectional curvature K to obtain an evaluation of $I(h)$ as in the following Proposition,

Proposition 1. *If K is δ -pinched, that is,*

$$\delta \leq K \leq 1$$

then with a constant c , $I(h)$ satisfies the following:

$$(0 \leq) I(h) - 1 \leq c\sqrt{1 - \delta}.$$

Therefore making $(1 - \delta)$ so small as to satisfy

$$c\sqrt{1 - \delta} < \varepsilon,$$

we get a diffeomorphism between M and (the standard) S^n .

Theorem 1. *If a compact, connected, simply connected Riemannian manifold M is δ -pinched with*

$$1 - (\varepsilon/c)^2 < \delta$$

then M is diffeomorphic to the standard n -sphere.

Unfortunately, our evaluation itself is not as good as that of D. Gromoll [2], though our method might allow to generalize the pinching problem and make it possible to treat the problem from an interesting point of view.

1. Preliminary remarks

Lemma 1. *Let h be a homeomorphism between complete Riemannian manifolds $M_l (l=1, 2)$, with metrics $d_l (l=1, 2)$ and let $\{U_i\}$ be an open covering of M_1 . Then if h satisfies on each open set U_i the following inequality;*

$$d_1(x, y)/k \leq d_2(h(x), h(y)) \leq kd_1(x, y) \quad (x, y \in U_i),$$

we have

$$I(h) \leq k.$$

Proof. For two points $p, q \in M_1$, take the minimizing geodesic $g(t)$ from p to q . It is possible to choose $t_j (j=0, \dots, N)$ such that the geodesic segment $g([t_{j-1}, t_j])$ lies completely in one of open sets U_i .

Therefore we have,

$$\begin{aligned} d_2(h(p), h(q)) &\leq \sum_j d_2(h(g(t_{j-1})), h(g(t_j))) \\ &\leq \sum_j kd_1(g(t_{j-1}), g(t_j)) \\ &\leq kd_1(p, q). \end{aligned}$$

Also we have in quite a similar way (just replacing h by h^{-1}) that

$$d_2(h(p), h(q)) \geq d_1(p, q)/k,$$

finishing the proof.

The condition that U_i is open may be replaced by an assurance that the subdivision of a geodesic segment by U_i consist only of finite segments. Therefore we get the following version of Lemma 1:

Corollary 1. *Let $(K_1, f), (K_2, g)$ be differentiable triangulations of M_1, M_2 , respectively, and assume that h satisfies the following 1), 2).*

- 1) $d_2(h(p), h(q)) \leq kd_1(p, q)$, for any p, q of each n -simplex Δ_1 of K_1 .
- 2) $d_1(h^{-1}(p), h^{-1}(q)) \leq kd_2(p, q)$, for any p, q of each n -simplex Δ_2 of K_2 .

Then we have

$$I(h) \leq k.$$

Lemma 2. *Suppose that there exist coordinate systems $\{U_i, f_i\}, \{U_i, g_i\}$ on M_1, M_2 , having the same Euclidean open sets U_i as local parameter systems, and*

that the homeomorphism h is given by $g_i \cdot f_i^{-1}$ on each open set $f_i(U_i)$. Then if the line elements ds_1, ds_2 (written in the parameter system of U_i) satisfy that

$$ds_1/k \leq ds_2 \leq kds_1,$$

we also have

$$I(h) \leq k.$$

Corollary 2. If h is piecewise differentiable on differentiable triangulations $(K, f), (K, g)$ of M_1, M_2 , then Lemma 2 holds when h is given by $g \cdot f^{-1}$ on each n -simplex Δ of K and the line elements ds_1, ds_2 (written in the coordinate of Δ) satisfy

$$ds_1/k \leq ds_2 \leq kds_1 \quad \text{on each } \Delta \in K.$$

2. The computation of $I(h)$

For a $1/4$ -pinched compact simply connected Riemannian manifold, the following facts i), ii) are known in [1, 4, 6, 8].

i) There are points $p, q \in M$ and a positive a , satisfying $\pi/2\sqrt{\delta} \leq a \leq \pi$ with $\delta = \min K$ such that

1) The open sets $U, V \subset M$ defined by

$$U = \{x \in M / d(x, p) < a\}, \quad V = \{y \in M / d(y, q) < a\}$$

cover M , that is, $U \cup V = M$.

2) The exponential maps defined at $p, q \in M$ send the open balls $\{X \in T_p(M) / |X| < a\}, \{Y \in T_q(M) / |Y| < a\}$ diffeomorphically onto U and V , respectively.

ii) Let N be a point set defined by

$$N = \{x \in M / d(x, p) = d(x, q)\},$$

then N possesses the following properties:

1) N is a differentiable submanifold of M and lies in $U \cap V$.

2) For every $x \in N$ there are a unique minimizing geodesic from p to x and a unique minimizing geodesic from q to x , we denote the initial directions of these geodesics by $g_+(x) \in T_p(M), g_-(x) \in T_q(M)$, respectively.

3) Every geodesic segment of length a starting at p of initial direction X cuts N exactly at one point which we denote by $f_+(X)$. Also every geodesic segment of length a starting at q of initial direction Y cuts N exactly at one point which we denote by $f_-(Y)$.

Using the facts i), ii), a homeomorphism h of the standard unit n -sphere S^n onto M is constructed through following steps a)-e):

a) Let P, Q be the north pole and the south pole of S^n and express a point

x of the northern hemi-sphere E_+ by the standard polar coordinate system at P :

$$x = (G_+(x), R_+(x)), G_+(x) \in T_P(S^n), 0 \leq R_+(x) \leq \pi/2.$$

Also write a point y in the southern hemi-sphere E_- by the polar coordinate system at Q :

$$y = (G_-(y), R_-(y)), G_-(y) \in T_Q(S^n), 0 \leq R_-(y) \leq \pi/2.$$

b) For a direction $X \in T_P(S^n)$, denote by $F_+(X)$ the point in the equator E at which the geodesic segment of initial direction X crosses E :

$$F_+(X) = (X, \pi/2).$$

Also define $F_-(Y)$ ($Y \in T_Q(S^n)$) to be the point in E at which the geodesic segment of initial direction y cuts E :

$$F_-(Y) = (Y, \pi/2).$$

c) Take a linear isometry α of $T_P(S^n)$ onto $T_P(M)$ and define an one to one map β of $T_Q(S^n)$ onto $T_Q(M)$ by

$$\beta(Y) = \begin{cases} g_- \circ f_+ \circ \alpha \circ G_+ \circ F_-(Y), & \text{if } |Y| = 1 \\ |Y| \beta(Y/|Y|) & \text{otherwise.} \end{cases}$$

d) Define an one to one map γ_+ of $T_P(M)$ onto itself by

$$\gamma_+(X) = 2 \operatorname{dist}(p, f_+(X/|X|)) X / \pi.$$

also define $\gamma_-(Y)$ on $T_Q(M)$ by

$$\gamma_-(Y) = 2 \operatorname{dist}(p, f_-(Y/|Y|)) Y / \pi.$$

e) Now the homeomorphism h of S^n onto M is given by

$$h(x) = \begin{cases} \exp(p) \circ \gamma_+ \circ \alpha \circ \exp(P)^{-1}(x), & \text{if } x \in E_+ \\ \exp(q) \circ \gamma_- \circ \beta \circ \exp(Q)^{-1}(x), & \text{if } x \in E_- . \end{cases}$$

In order to prove that h is approximated by diffeomorphisms if the sectional curvature K of M is sufficiently pinched, we evaluate $I(h)$ relative to the standard metric on S^n and the given Riemannian metric on M . The evaluation is done through the three steps: first we evaluate $I(\exp(p) \circ \alpha \circ \exp(P)^{-1})$, next $I(\exp(p) \circ \gamma_+ \circ \exp(p)^{-1})$ and $I(\exp(q) \circ \gamma_- \circ \exp(q)^{-1})$, and finally we evaluate $I(\exp(q) \circ \beta \circ \exp(Q)^{-1})$. Since in general we know that

$$I(A \circ B) \leq I(A) I(B)$$

for any maps A, B , these three steps complete our evaluation.

2.1 First step, on $I(\exp(p) \circ \alpha \circ \exp(P)^{-1})^*$.

Take orthogonal directions $X, Y \in T_p(M)$, then because of i) 2), we may apply Rauch's comparison theorem to the arc $c(\theta) = r(X \cos \theta + Y \sin \theta)$ ($t_1 \leq \theta \leq t_2$, $0 \leq r \leq \pi/2$) and to S^n, M, α . We get that

$$L(\exp(P) \circ \alpha^{-1} \circ c) \leq L(\exp(p) \circ c),$$

where $L(\varphi)$ denotes the length of the arc φ .

Let $S^n(\delta)$ be the sphere of constant curvature δ (δ is the positive pinching of the sectional curvature K of M from below; $\delta \leq K \leq 1$), then we also can apply Rauch's theorem to $c(\theta)$, M , $S^n(\delta)$ and a linear isometry α' of $T_p(S^n(\delta))$ onto $T_p(M)$, to get that

$$L(\exp(p) \circ c) \leq L(\exp(P) \circ \alpha'^{-1} \circ c)$$

Since it is elementary to show that

$$\begin{aligned} L(\exp(P) \circ \alpha^{-1} \circ c) &= (t_2 - t_1) \sin r \\ L(\exp(P) \circ \alpha'^{-1} \circ c) &= \frac{(t_2 - t_1)}{\sqrt{\delta}} \sin \sqrt{\delta} r, \end{aligned}$$

we deduce that

$$\sin r \leq \frac{L(\exp(p) \circ c)}{t_2 - t_1} \leq \frac{1}{\sqrt{\delta}} \sin \sqrt{\delta} r.$$

In order to have an evaluation of the ratio of the line elements on S^n and M , consider the submanifold $M(X, Y)$ of M consisting of elements of the form $\exp(p)(rX \cos \theta + rY \sin \theta)$ and parametrize the plane of X, Y by (r, θ) . The line element of M restricted on $M(X, Y)$, then, is written in the form $dr^2 + \mu^2(r, \theta)d\theta^2$. Since the function $\mu(r, \theta)$ is nothing but the limit of $L(\exp(p) \circ c)/(t_2 - t_1)$ when $t_1 \rightarrow \theta$, the above inequality yields that

$$dr^2 + \sin^2 r d\theta^2 \leq dr^2 + \mu^2(r, \theta) d\theta^2 \leq dr^2 + \frac{1}{\delta} \sin^2 \sqrt{\delta} r d\theta^2.$$

Therefore we get that for any $X, Y \in T_p(M)$ it holds that

$$dr^2 + \sin^2 r d\theta^2 \leq dr^2 + \mu^2(r, \theta) d\theta^2 \leq \frac{1}{\delta} (dr^2 + \sin^2 r d\theta^2),$$

on $M(X, Y)$. Thus we may conclude that

$$I(\exp(p) \circ \alpha \circ \exp(P)^{-1}) \leq 1/\sqrt{\delta},$$

by virtue of Lemma 2.

* The description in section 2.1, is due to professor Y. Tsukamoto and improves the author's original (less complete) one.

2.2 Second step, on $I(\exp(q) \circ \gamma_+ \circ \exp(q)^{-1})$

The following fact iii) also is known for a compact connected δ -pinched simply connected manifold $M(\delta > 1/4)$,

- iii) 1) $\pi \leq \text{diam}(M) \leq \pi/\sqrt{\delta}$
 2) Let p, q be the points in i) 2), then for any $x \in M$,
 $d(p, x) \leq \pi/2\sqrt{\delta}$ or $d(q, x) \leq \pi/2\sqrt{\delta}$.

In order to evaluate $I(\exp(q) \circ \gamma_+ \circ \exp(q)^{-1})$, we first consider the differential in X of the function λ defined by

$$\lambda(X) = |\gamma_+(X)|/|X|.$$

Take $x, y \in N$ and let $\triangle PAB, \triangle QA'B'$ be triangles in euclidean space such that

$$\begin{aligned} d(p, x) &= d(P, A), & d(p, y) &= d(P, B), & d(x, y) &= d(A, B) \\ d(q, x) &= d(Q, A'), & d(q, y) &= d(Q, B'), & d(x, y) &= d(A', B'). \end{aligned}$$

Suppose $\angle A' \leq \angle B'$ for instance, in $\triangle QA'B'$, we then have by Toponogov's comparison theorem that

$$\pi - \angle Q = \angle A' + \angle B' \leq 2\angle B' \leq 2\angle qyx.$$

Since, in general, it holds that

$$\angle qyx + \angle xyp + \angle pyq \leq 2\pi,$$

we get that

$$\pi/2 - \angle P/2 \leq \angle PBA \leq \angle pyx \leq \pi/2 + \angle P/2 + (\pi - \angle pyq),$$

hence we see that in $\triangle PBA$

$$\pi/2 - 3\angle P/2 - (\pi - \angle pyq) \leq \angle PAB = \angle QA'B' \leq \pi/2 - \angle P/2.$$

Therefore we have that

$$\begin{aligned} |d(p, x) - d(p, y)| &= |d(P, A) - d(P, B)| \\ &\leq d(P, A) \left| \frac{2 \sin \angle P/2 \sin (\angle PAB/2 - \angle PBA/2)}{\sin \angle PBA} \right| \\ &\leq \pi |\sin \angle P/2 \tan (\angle P + (\pi - \angle pyq))|/\sqrt{\delta}. \end{aligned}$$

Let now $x = f_+(X)$, $y = f_+(X + dX)$, then the inequality above yields that

$$|\gamma'(X)| \leq \tan \angle pyq/\sqrt{\delta}$$

On the other hand, Toponogov's theorem applied to the geodesic triangle $\triangle pyq$, on which

$$\pi \leq d(p, q), \quad d(p, y) = d(q, y) \leq \pi/2\sqrt{\delta},$$

yields that

$$\cos \angle pyq \leq 1 - d^2(p, q)/2d^2(p, y) \leq 1 - 2\delta.$$

Thus we get that

$$|\gamma'(X)| \leq 4\sqrt{1-\delta}.$$

Since the homeomorphism $\exp(p) \circ \gamma_+ \circ \exp(p)^{-1}$ leaves the submanifold $M(X, Y)$ of (2, 1) for orthogonal directions X, Y invariant, we may evaluate the effect of $(\exp(p) \circ \gamma_+ \circ \exp(p)^{-1})^*$ on the line element ds of $M(X, Y)$, in order to get an evaluation of $I(\exp(p) \circ \gamma_+ \circ \exp(p)^{-1})$. We compare two quadratic forms $I(x, y)$, $I_0(x, y)$ given by

$$\begin{aligned} I(x, y) &= (\lambda(\theta)x)^2 + 2\lambda'(\theta)\lambda(\theta)rx + \{\mu^2(\theta, \lambda(\theta)r) + (\lambda'(\theta)r)^2\}y^2, \\ I_0(x, y) &= x^2 + \mu^2(\theta, r)y, \end{aligned}$$

to get the following: If a positive k satisfies that

- 1) $\lambda^2(\theta) \leq k \leq 4$
- 2) $4(\lambda'(\theta)r)^2 \leq \mu^2(\theta, r) \left(k - \left(\frac{\mu(\theta, \lambda(\theta)r)}{\mu(\theta, r)} \right)^2 \right) (k - \lambda^2),$

then the quadratic form $kI_0(x, y)$ dominates $I(x, y)$, that is,

$$I(x, y) \leq kI_0(x, y) \quad \text{for any } x, y.$$

Since we have that

$$\delta \sin r \leq \mu(r, \theta) \leq \frac{1}{\delta} \sin r, \quad 1 \leq \lambda \leq 1/\sqrt{\delta}$$

from (2. 1) and from iii) 1), 2), we see that the condition 2) above is fulfilled with k such that

$$k \geq (1 + 4\pi\delta^2\sqrt{1-\delta})/\delta^3$$

Thus we have that, if $\delta \geq 99/100$ e.g., then with $k_1 = (1 + 4\pi\delta^2\sqrt{1-\delta})/\delta^3$, it holds that

$$(\exp(p) \circ \gamma_+ \circ \exp(p)^{-1})^* ds \leq k_1 ds.$$

Quite similarly, we also have that with $k_2 = \delta(1 - 4\pi\sqrt{1-\delta})$, it holds that

$$(\exp(p) \circ \gamma_+ \circ \exp(p)^{-1})^* ds \geq k_2 ds.$$

Thus we may conclude that

$$I(\exp(p) \circ \gamma_+ \circ \exp(p)^{-1}) \leq k_0,$$

where $k_0 = \max(k_1, 1/k_2)$.

As in the same way above, we get that

$$I(\exp(q) \circ \gamma_- \circ \exp(q)^{-1}) \leq k_0.$$

2.3 Third step, on $I(\exp(q) \circ \beta \circ \exp(Q)^{-1})$.

We take two points x, y in E_- with polar coordinates (X, r) (Y, r) ($X, Y \in T_Q(S''), 0 \leq r \leq \pi/2$). Apply the evaluation in 2. 1 and 2. 2 to points $F_-(X), F_-(Y) \in E$, where two maps $h_+ = \exp(p) \circ \gamma_+ \circ \alpha \circ \exp(P)^{-1}$ and $h_- = \exp(q) \circ \gamma_- \circ \beta \circ \exp(Q)^{-1}$ coincide, to get that

$$\delta/k_0 \leq \frac{d(h_- \circ F_-(X), h_- \circ F_-(Y))}{d(F_-(X), F_-(Y))} \leq k_0/\delta.$$

On the other hand, Rauch's theorem applied to a linear isometry $\tilde{\beta}$ of $T_Q(S'')$ onto $T_Q(M)$ and to S'' (or $S''(\delta)$), M yields that

$$1 \leq \frac{d(\exp(q) \circ \tilde{\beta} \circ \exp(Q)^{-1}(a), \exp(q) \circ \tilde{\beta} \circ \exp(Q)^{-1}(b))}{d(a, b)} \leq \frac{1}{\sqrt{\delta}}$$

for $a, b \in E_-$. Let $\xi = \tilde{\beta}^{-1}\beta(X)$, $\eta = \tilde{\beta}^{-1}\beta(Y)$, then we have that

$$1 \leq \frac{d(\tilde{h} \exp(Q)(sX), \exp \tilde{h}(Q)(sY))}{d(\exp(Q)(s\xi), \exp(Q)(s\eta))} \leq \frac{1}{\sqrt{\delta}},$$

where $\tilde{h} = \exp(q) \circ \beta \circ \exp(Q)^{-1}$. Substitute s by $\pi/2$ and by r in the inequality above to have that

$$\sqrt{\delta} \leq \frac{d(\exp(Q)(r\xi), \exp(Q)(r\eta))}{d(F_-(\xi), F_-(\eta))} \cdot \frac{d(\tilde{h} \circ F_-(X), \tilde{h} \circ F_-(Y))}{d(\tilde{h}(x), \tilde{h}(y))} \leq \frac{1}{\sqrt{\delta}}.$$

Since the ratio

$$\frac{d(\exp(Q)(r\xi), \exp(Q)(r(\xi + d\xi)))}{d(F_-(\xi), F_-(\xi + d\xi))} = \sin r$$

depends only on r , we get that if Y is sufficiently near to X , then

$$\frac{\delta}{k_0} \leq \frac{d(\tilde{h}(x), \tilde{h}(y))}{d(\tilde{h} \circ F_-(X), \tilde{h} \circ F_-(Y))} \cdot \frac{d(H \circ \tilde{h} \circ F_-(X), H \circ \tilde{h} \circ F_-(Y))}{d(x, y)} \leq \frac{k_0}{\delta},$$

where $H = \exp(q) \circ \gamma_- \circ \exp(q)^{-1}$. Combining this with the result of 2. 2, we have that

$$\frac{\delta}{k_0^2} \leq \frac{d(\tilde{h}(x), \tilde{h}(y))}{d(x, y)} \leq \frac{k_0^2}{\delta}$$

Thus we conclude that

$$I(\tilde{h}) \leq k_0^2/\delta$$

because \tilde{h} preserves length along longitude.

Consequently we get that

$$I(h_-) \leq I(H) \cdot I(\tilde{h}) \leq k_0^3/\delta$$

Therefore we finally have that

$$I(h) \leq k_0^3/\delta,$$

from Corollary 1, finishing the proof of Proposition 1 at the beginning.

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